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**Textbook Multigrid Efficiency for
Computational Fluid Dynamics
Simulations**

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Introduction

Considerable progress over the past thirty years has been made in the development of large-scale computational fluid dynamics (CFD) solvers for the Euler and Navier-Stokes equations. Computations are used routinely to design the cruise shapes of transport aircraft through complex-geometry simulations involving the solution of 25-100 million equations; in this arena, the number of wind-tunnel tests for a new design has been substantially reduced. However, simulations of the entire flight envelope of the vehicle, including maximum lift, buffet onset, flutter, and control effectiveness, have not been as successful in eliminating the reliance on wind-tunnel testing. These simulations involve unsteady flows with more separation and stronger shock waves than at cruise. The main reasons limiting further inroads of CFD into the design process are: (1) the reliability of turbulence models and (2) the time and expense of the numerical simulation. Because of the prohibitive resolution requirements of direct simulations at high Reynolds numbers, transition and turbulence modeling is expected to remain an issue for the near term.¹ The focus of this paper addresses the latter problem by attempting to attain optimal

efficiencies in solving the governing equations. Typically current CFD codes based on the use of multigrid acceleration techniques and multistage Runge-Kutta time-stepping schemes are able to converge lift and drag values for cruise configurations within approximately 1000 residual evaluations. More complexity in the geometry or physics generally requires many more residual evaluations to converge, and sometimes convergence cannot be attained. An optimally convergent method is defined²⁻⁵ as having textbook multigrid efficiency (TME), meaning the solutions to the governing system of equations are attained in a computational work which is a small (less than 10) multiple of the operation count in the discretized system of equations (residual evaluations). Thus, there is a potential gain of more than two orders of magnitude in operation count reduction if TME could be attained.

In this paper, a distributed relaxation approach to achieving TME for Reynolds-averaged Navier-Stokes (RANS) equations is discussed along with the foundations that form the basis of this approach. Because the governing equations are a set of coupled nonlinear conservation equations with discontinuities (shocks, slip lines, etc.) and singularities (flow- or grid-induced), the difficulties are many. The TME methodology insists that each of the difficulties should be isolated, analyzed, and solved systematically using a carefully constructed series of model problems. An important aspect of the distributed relaxation approach is a separate treatment of each of the factors (elliptic and hyperbolic) constituting the system of partial differential equations. Another distinguishing aspect of the approach is that these factors are treated directly for steady-state flows rather than through pseudo-time marching methods; time-dependent flow solvers can be constructed within this approach and in principle are simpler to develop than steady-state solvers. An extensive list of envisioned difficulties in attaining TME

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for CFD simulations, along with possible solutions, are discussed elsewhere.^{4,5} This paper also summarizes recent progress towards the attainment of TME in basic CFD simulations.

Foundations for Textbook Multigrid Efficiency

The basic framework for TME solvers is full multigrid (FMG) algorithms.^{2,3,6-9} In FMG algorithms, the solution process is started on a very coarse grid where the computational cost of solution is negligible. The coarse-grid solution is then interpolated to the next fine grid to form an initial approximation. Few multigrid full approximation scheme (FAS) cycles, or possibly just one, are performed next to obtain an improved fine-grid solution approximation. Then, the process proceeds to finer grids until the solution on the target finest grid is achieved.

In solution of highly nonlinear problems, a good initial guess is important. A general way to obtain such an initial guess is by continuation, in which the solution to the target problem is approached through the solutions of a sequence of parameterized problems. Usually the problem starting the continuation process is easy to solve, and difficulty gradually increases with control parameter approaching the target value; this continuation process can often be integrated into an FMG solver. For example, with viscosity as the control parameter, at the coarse grids more artificial viscosity can be used, then gradually be taken out as the algorithm proceeds to finer levels. Such FMG continuation is often natural because larger numerical viscosity is introduced on coarse grids, even without aiming at continuation.

A version named λ -FMG algorithm provides the device needed for optimal adaptive local refinement. Efficient multigrid solvers based on this approach have been demonstrated.¹⁰

The objective of FMG algorithms (and TME methods in particular) is fast convergence to the solution of the differential equations, not necessarily fast asymptotic residual convergence. The natural solution tolerance is the discretization error defined as the difference between the exact solutions of discrete and differential problems. Thus, the quality of a solution approximation on a given grid can be measured by the relative magnitude of algebraic errors in comparison with the discretization error level. The algebraic error is defined as the difference between the exact and approximate solutions of the discrete problem. On any grid in an FMG algorithm, we expect the algebraic errors after few multigrid cycles to be always less than the discretization error.

On the other hand, a fast residual convergence is considered as an important monitoring tool. In many practical cases, it is possible to develop a solver exhibiting fast residual convergence rates without com-

promising TME. Note however that sometimes the quality of the target-grid solution can be much improved by double discretization methods applying for relaxation a different scheme than that used in calculating residuals transferred to the coarse grid; zero target-grid residuals might not be the aim in this case.

Standard multigrid methods efficient for elliptic problems separate the treatment of oscillatory and smooth error components. The former are efficiently reduced in single-grid iterations (relaxation); the latter are well approximated on coarse grids and, hence, eliminated through the coarse-grid correction. The difficulties associated with extending TME for solution of the RANS equations relate to the fact that these equations are a system of coupled nonlinear equations that is not, even for subsonic Mach numbers, fully elliptic, but contain hyperbolic partitions. The efficiency of classical multigrid methods severely degrades for nonelliptic problems because some smooth *characteristic components* cannot be adequately approximated on coarse grids.¹¹⁻¹³ The characteristic components are much smoother in the characteristic directions than in other directions. To be efficient, a multigrid solver for nonelliptic problems has to adequately address three types of errors: (1) high-frequency error components, (2) uniformly smooth error components, (3) characteristic error components.

If the target discretization is strongly *h-elliptic* (or *semi-h-elliptic*) one can design a local (or block-wise) relaxation procedure efficiently reducing all high-frequency error components. By definition,^{2,3,8,11} a discrete scalar (not necessarily elliptic) operator $L[u]$ possesses a good measure of *h-ellipticity*, if the absolute value of its symbol $|L(\bar{\theta})| = |e^{-i(\bar{\theta} \cdot \mathbf{j})} L[e^{i(\bar{\theta} \cdot \mathbf{j})}]|$ is well separated from zero for all high-frequency Fourier modes. Here $\mathbf{j} = (j_x, j_y, j_z)$ are the grid indexes and $\bar{\theta} = (\theta_x, \theta_y, \theta_z)$, $0 \leq |\theta_x|, |\theta_y|, |\theta_z| \leq \pi$ are normalized Fourier frequencies. High-frequency Fourier modes are the modes satisfying $\max(|\theta_x|, |\theta_y|, |\theta_z|) \geq \frac{\pi}{2}$. For systems, the measure of *h-ellipticity* is defined as the measure of the determinant operator.

Coarse-grid correction is usually efficient for uniformly smooth error components. An effective reduction of characteristic error components can be achieved either by designing a proper relaxation scheme reducing not only high-frequency but smooth error components as well (which can be done in many non-uniformly-elliptic cases by downstream ordering of relaxation steps^{11,12}) or by adjusting coarse-grid operators for a better characteristic-component approximation.

Multigrid methods efficiently reducing all the three aforementioned types of error have been developed for *scalar* nonelliptic operators.¹³⁻¹⁷ Similar efficiency for solution of the RANS system of differential equations can be achieved by exploiting the system factorizability. *Factorizability* is a property of the system deter-

minant to be factored as a product of simple scalar factors representing the (elliptic and hyperbolic) partitions contributing to the target differential system.

For the subsonic compressible Euler equations, the first TME solvers exploiting factorizability of the system have been developed by Ta'asan.¹⁸⁻²⁰ New *canonical variables* have been introduced, and in these variables, the Euler system of equations has a block upper triangular form with the main diagonal blocks consisting of the basic components of the system.²¹ The main disadvantage of this formulation is that it is not easily generalized to viscous and unsteady problems, especially in three dimensions.

Another approach toward achieving TME for solution of the Euler and incompressible Navier-Stokes equations²²⁻²⁴ is based on the pressure-equation formulation which effectively separates elliptic and hyperbolic factors of the system. This formulation has been extended to generalized coordinates.^{25,26} Grid-independent convergence rates have been demonstrated for inviscid flow around airfoils and for viscous incompressible flow past a finite flat plate and a non-lifting airfoil. This approach has met difficulties in generalizing to viscous compressible flows.

A more general approach is the *distributed relaxation* method.^{2,3,12,27-29} The details of the method are presented in the next section. An important feature of this approach is that the separation of different factors constituting the system determinant occurs only in computing updates at the relaxation stage of an outer multigrid solver; the original coupled (conservative) equations are always used to compute residuals. This feature allows a lot of freedom in relaxation scheme design since different schemes may be applied to different flow regions. The distributed relaxation approach reduces the problem of relaxing a complicated system of discretized coupled differential equations to relaxation of scalar factors constituting the system determinant. Note that relaxation schemes for the scalar factors may include separate multigrid solvers. Usually, distributed relaxation can be applied throughout the entire domain having the full effect away from discontinuities (shocks, slip lines) in the regular (smoothly varying) flow field. Some local relaxation sweeps should be applied in these special regions after (and perhaps also before) the distributed relaxation pass to reduce residuals. The general rule for efficient adaptive relaxation is to apply additional relaxation sweeps wherever local residuals significantly exceed the average level characterizing the neighboring regular flow field.

The distributed relaxation scheme design for the RANS system of equations can be significantly simplified if the target *discretization* is also factorizable, i.e., the discrete system determinant can be represented as a product of discrete scalar factors, each of them approximating a corresponding factor of the determinant

of the *differential* RANS equations. In fact, since distributed relaxation is applied only for solution updates in a relaxation sweep, the factorizability property is only required for the *principal linearization* operator. The principal linearization of a *scalar* equation contains the linearization terms that make a major contribution to the residual per a unit change. The principal terms thus generally depend on the scale, or mesh size, of interest. For example, the discretized highest derivative terms are principal on grids with small enough mesh size. For a discretized *system* of differential equations, the principal terms are those that contribute to the principal terms of the system *determinant*. If the principal linearization is discretely factorizable and efficient relaxation schemes for the corresponding discrete scalar factors are available, distributed relaxation efficiently reduces high-frequency and characteristic error components as well.

For nonfactorizable (but *h*-elliptic) discretizations of the RANS equations, the general scheme for relaxation updates should include two different passes: (1) Direct relaxation of the target discrete scheme that is efficient for high-frequency error reduction and (2) distributed relaxation based on reasonable discretizations of the scalar factors of the differential system determinant (these discretizations are not derived from the target discrete system) that eliminates characteristic error components.

Distributed Relaxation

The system of time-dependent compressible Navier-Stokes equations can be written as

$$\partial_t \mathbf{Q} + \mathbf{R}(\mathbf{Q}) = 0, \quad (1)$$

where the conserved variables are $\mathbf{Q} \equiv (\rho u, \rho v, \rho w, \rho, \rho E)^T$, representing the momentum vector, density, and total energy per unit volume, and $\mathbf{R}(\mathbf{Q})$ is a spatial divergence of a vector function representing convection and viscous and heat transfer effects. In general, the simplest form of the differential equations corresponds to nonconservative equations expressed in primitive variables, here taken as the set composed of velocity, pressure, and internal energy, $\mathbf{q} = (u, v, w, p, \epsilon)^T$. For a perfect gas, the primitive and conservative variables are connected through the following relations

$$\begin{aligned} p &= (\gamma - 1)\rho\epsilon, \\ \epsilon &= E - \frac{1}{2}(u^2 + v^2 + w^2), \\ c^2 &= \gamma p / \rho, \end{aligned}$$

where c is the speed of sound and γ is the ratio of specific heats.

The time-dependent nonconservative equations are found readily by transforming the time-dependent conservative equations.

$$\begin{aligned}\frac{\partial \mathbf{q}}{\partial \mathbf{Q}} [\partial_t \mathbf{Q} + \mathbf{R}] &= 0, \\ \partial_t \mathbf{q} + \frac{\partial \mathbf{q}}{\partial \mathbf{Q}} \mathbf{R} &= 0,\end{aligned}$$

where $\frac{\partial \mathbf{q}}{\partial \mathbf{Q}}$ is the Jacobian matrix of the transformation. For steady-state equations, the time derivative is dropped. In an iterative procedure, the correction $\delta \mathbf{q} \equiv \mathbf{q}^{n+1} - \mathbf{q}^n$, where n is an iteration counter, can be computed from the equation

$$\mathbf{L} \delta \mathbf{q} = -\frac{\partial \mathbf{q}}{\partial \mathbf{Q}} \mathbf{R}, \quad (2)$$

where \mathbf{L} is a linear operator containing the viscous and inviscid terms of the nonconservative operator that are principal at the scale h . Thus, a good correction is expected away from discontinuities in the regular (smoothly varying) flow field.

Usually, the principle linearization operator \mathbf{L} is derived under the assumption of solution smoothness that requires the magnitudes of all solution differences to be smaller than the solution magnitude. The approximation inherent in this principal linearization can be illustrated by using the nonlinear convection equation,

$$N(u) \equiv u \partial_x^h u = f.$$

A full linearization for a correction δu results in

$$\frac{\partial N}{\partial u} \delta u \equiv \delta u \partial_x^h u + u \partial_x^h \delta u = f - N(u).$$

The principal linearization of this correction equation at scale h is

$$u \partial_x^h \delta u = f - N(u),$$

where the term $\delta u \partial_x^h u$ can be neglected as $h \rightarrow 0$ assuming that $\partial_x^h u$ is bounded. This approximation is also termed Picard iteration, which is exact for the linear case. Note that on coarser grids, the term $\delta u \partial_x^h u$ may not be so small. The FMG algorithm plays a very important role in preventing fine-grid initial approximations with large high-frequency algebraic errors violating the smoothness assumption.

While significantly simplified by retaining only principal terms, the system (2) is still a set of coupled equations containing elliptic and hyperbolic components. Therefore, collective Gauss-Seidel relaxation of \mathbf{L} is not often effective, and factorizability of \mathbf{L} must be exploited. The distributed relaxation method replaces $\delta \mathbf{q}$ in (2) by $\mathbf{M} \delta \mathbf{w}$, so that the resulting matrix $\mathbf{L} \mathbf{M}$ becomes lower triangular. The diagonal elements of $\mathbf{L} \mathbf{M}$ are composed ideally of the separable factors of the matrix \mathbf{L} determinant. These factors are scalar differential operators of first or second order, so their efficient relaxation is a much simpler task than relaxing

the entire system associated with \mathbf{L} . In relaxing scalar factors, the changes introduced in the "ghost" variables $\delta \mathbf{w}$ (the variables $\delta \mathbf{w}$ are "ghost" because they need not be explicitly used in computations) during relaxation are distributed, with the pattern of distribution matrix \mathbf{M} , to the primitive variables. To obtain the optimal (textbook) efficiency, relaxation of each factor should incorporate the essential part of an efficient multigrid solver for its corresponding operator: sometimes this essential part is just the relaxation part of that solver, sometimes this may even be an entire separate multigrid solver applied at some proper subdomains.

Incompressible Navier-Stokes Equations

The steady-state incompressible Navier-Stokes equations can be written as

$$\begin{aligned}Q_\nu \mathbf{u} + \nabla p &= 0, \\ \nabla \cdot \mathbf{u} &= 0,\end{aligned}$$

where $\mathbf{u} = (u, v, w)^T$ is the velocity vector and $Q_\nu = \mathbf{u} \cdot \nabla - \nu \Delta$ is a convection-diffusion operator. ($Q = Q_0$ denotes the particular case with zero ($\nu = 0$) physical diffusion.) Under the solution smoothness assumption the principal linearization operator is given by

$$\mathbf{L} = \begin{bmatrix} Q_\nu & 0 & 0 & \partial_x \\ 0 & Q_\nu & 0 & \partial_y \\ 0 & 0 & Q_\nu & \partial_z \\ \partial_x & \partial_y & \partial_z & 0 \end{bmatrix}, \quad (3)$$

$$\det \mathbf{L} = -Q_\nu^2 \Delta, \quad (4)$$

where the coefficients (u, v, w) in Q_ν are computed from the previous solution approximation and fixed during each distributed relaxation step. An appropriate matrix \mathbf{M} is

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & -\partial_x \\ 0 & 1 & 0 & -\partial_y \\ 0 & 0 & 1 & -\partial_z \\ 0 & 0 & 0 & Q_\nu \end{bmatrix}, \quad (5)$$

yielding the lower triangular operator

$$\mathbf{L} \mathbf{M} = \begin{bmatrix} Q_\nu & 0 & 0 & 0 \\ 0 & Q_\nu & 0 & 0 \\ 0 & 0 & Q_\nu & 0 \\ \partial_x & \partial_y & \partial_z & -\Delta \end{bmatrix}. \quad (6)$$

Euler Equations

The conservation form for the Euler equations is given by (1) with

$$\mathbf{R}(\mathbf{Q}) \equiv \partial_x \mathbf{F}(\mathbf{Q}) + \partial_y \mathbf{G}(\mathbf{Q}) + \partial_z \mathbf{H}(\mathbf{Q}), \quad (7)$$

$$\mathbf{F}(\mathbf{Q}) = \begin{pmatrix} \rho u^2 + p \\ \rho uv \\ \rho uw \\ \rho u \\ \rho u E + up \end{pmatrix}, \quad \mathbf{G}(\mathbf{Q}) = \begin{pmatrix} \rho uv \\ \rho v^2 + p \\ \rho vw \\ \rho v \\ \rho v E + vp \end{pmatrix},$$

$$\mathbf{H}(\mathbf{Q}) = \begin{pmatrix} \rho uw \\ \rho vw \\ \rho w^2 + p \\ \rho w \\ \rho w E + wp \end{pmatrix}.$$

The Jacobian matrix of the conservative-to-nonconservative transformation is defined as

$$\frac{\partial \mathbf{q}}{\partial \mathbf{Q}} = \begin{pmatrix} 1/\rho & 0 & 0 & 0 & 0 \\ 0 & 1/\rho & 0 & 0 & 0 \\ 0 & 0 & 1/\rho & 0 & 0 \\ 0 & 0 & 0 & \gamma - 1 & 0 \\ 0 & 0 & 0 & 0 & 1/\rho \end{pmatrix} \times$$

$$\begin{pmatrix} 1 & 0 & 0 & -u & 0 \\ 0 & 1 & 0 & -v & 0 \\ 0 & 0 & 1 & -w & 0 \\ -u & -v & -w & \frac{u^2+v^2+w^2}{2} & 1 \\ -u & -v & -w & -\epsilon + \frac{u^2+v^2+w^2}{2} & 1 \end{pmatrix}.$$

In the regions of smoothly varying solution, the principal linearization of the nonconservative operator is

$$\mathbf{L} = \begin{bmatrix} Q & 0 & 0 & \frac{1}{\rho} \partial_x & 0 \\ 0 & Q & 0 & \frac{1}{\rho} \partial_y & 0 \\ 0 & 0 & Q & \frac{1}{\rho} \partial_z & 0 \\ \rho c^2 \partial_x & \rho c^2 \partial_y & \rho c^2 \partial_z & Q & 0 \\ \frac{c^2}{\gamma} \partial_x & \frac{c^2}{\gamma} \partial_y & \frac{c^2}{\gamma} \partial_z & 0 & Q \end{bmatrix}. \quad (8)$$

The determinant of the matrix operator \mathbf{L} is

$$\det \mathbf{L} = Q^3 [Q^2 - c^2 \Delta], \quad (9)$$

where Δ is the Laplace operator, and $Q^2 - c^2 \Delta$ represents the full-potential operator.

A possible distribution matrix \mathbf{M} is given by

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & -\frac{1}{\rho} \partial_x & 0 \\ 0 & 1 & 0 & -\frac{1}{\rho} \partial_y & 0 \\ 0 & 0 & 1 & -\frac{1}{\rho} \partial_z & 0 \\ 0 & 0 & 0 & Q & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (10)$$

and

$$\mathbf{L}\mathbf{M} = \begin{bmatrix} Q & 0 & 0 & 0 & 0 \\ 0 & Q & 0 & 0 & 0 \\ 0 & 0 & Q & 0 & 0 \\ \rho c^2 \partial_x & \rho c^2 \partial_y & \rho c^2 \partial_z & Q^2 - c^2 \Delta & 0 \\ \frac{c^2}{\gamma} \partial_x & \frac{c^2}{\gamma} \partial_y & \frac{c^2}{\gamma} \partial_z & -\frac{c^2}{\rho \gamma} \Delta & Q \end{bmatrix}. \quad (11)$$

Compressible Navier-Stokes Equations

The conservative compressible Navier-Stokes equations are formulated in the form (1) with $\mathbf{R}(\mathbf{Q})$ defined in (7).

$$\mathbf{F}(\mathbf{Q}) = \begin{pmatrix} \rho u^2 + p - 2\mu \partial_x u - \lambda(\nabla \cdot \mathbf{u}) \\ \rho uv - \mu(\partial_x v + \partial_y u) \\ \rho uw - \mu(\partial_x w + \partial_z u) \\ \rho u \\ \rho u E + up - \lambda u(\nabla \cdot \mathbf{u}) - \mu \tau_1 - \kappa \partial_x \epsilon \end{pmatrix}.$$

$$\mathbf{G}(\mathbf{Q}) = \begin{pmatrix} \rho uv - \mu(\partial_x v + \partial_y u) \\ \rho v^2 + p - 2\mu \partial_y v - \lambda(\nabla \cdot \mathbf{u}) \\ \rho vw - \mu(\partial_y w + \partial_z v) \\ \rho v \\ \rho v E + vp - \lambda v(\nabla \cdot \mathbf{u}) - \mu \tau_2 - \kappa \partial_y \epsilon \end{pmatrix}.$$

$$\mathbf{H}(\mathbf{Q}) = \begin{pmatrix} \rho uw - \mu(\partial_x w + \partial_z u) \\ \rho vw - \mu(\partial_y w + \partial_z v) \\ \rho w^2 + p - 2\mu \partial_z w - \lambda(\nabla \cdot \mathbf{u}) \\ \rho w \\ \rho w E + wp - \lambda w(\nabla \cdot \mathbf{u}) - \mu \tau_3 - \kappa \partial_z \epsilon \end{pmatrix}.$$

where

$$\begin{aligned} \tau_1 &= 2u \partial_x u + v(\partial_x v + \partial_y u) + w(\partial_x w + \partial_z u), \\ \tau_2 &= 2v \partial_y v + u(\partial_x v + \partial_y u) + w(\partial_y w + \partial_z v), \\ \tau_3 &= 2w \partial_z w + u(\partial_x w + \partial_z u) + v(\partial_y w + \partial_z v). \end{aligned}$$

μ and λ are viscosity coefficients, and κ is the coefficient of heat conductivity.

The corresponding nonconservative formulation is given by

$$\begin{aligned} &\left((\mathbf{u} \cdot \nabla) - \frac{\mu}{\rho} \Delta - \frac{\lambda}{\rho} \partial_{xx} \right) u - \frac{\lambda}{\rho} (\partial_{xy} v + \partial_{xz} w) + \frac{1}{\rho} \partial_x p = 0, \\ &\left((\mathbf{u} \cdot \nabla) - \frac{\mu}{\rho} \Delta - \frac{\lambda}{\rho} \partial_{yy} \right) v - \frac{\lambda}{\rho} (\partial_{xy} u + \partial_{yz} w) + \frac{1}{\rho} \partial_y p = 0, \\ &\left((\mathbf{u} \cdot \nabla) - \frac{\mu}{\rho} \Delta - \frac{\lambda}{\rho} \partial_{zz} \right) w - \frac{\lambda}{\rho} (\partial_{xz} u + \partial_{yz} v) + \frac{1}{\rho} \partial_z p = 0, \\ &\rho c^2 (\nabla \cdot \mathbf{u}) + (\mathbf{u} \cdot \nabla) p + (\gamma - 1)(-\kappa \Delta \epsilon + \Phi) = 0, \\ &\frac{c^2}{\gamma} (\nabla \cdot \mathbf{u}) + (\mathbf{u} \cdot \nabla) p - \frac{\kappa}{\rho} \Delta \epsilon + \rho \Phi = 0, \end{aligned}$$

where

$$\begin{aligned} \Phi &\equiv \mu \left(2(\partial_x u)^2 + 2(\partial_y v)^2 + 2(\partial_z w)^2 \right. \\ &\quad \left. + (\partial_x v + \partial_y u)^2 + (\partial_x w + \partial_z u)^2 + (\partial_y w + \partial_z v)^2 \right) \\ &\quad + \lambda (\partial_x u + \partial_y v + \partial_z w)^2. \end{aligned}$$

Assuming solution smoothness and constant viscosity and heat conduction coefficients, the principal linearization operator \mathbf{L} , keeping the terms principal on both the viscous and inviscid scales, is given by

$$\mathbf{L} = \begin{bmatrix} Q_{\frac{\mu}{\rho}} - \frac{\lambda}{\rho} \partial_{xx} & -\frac{\lambda}{\rho} \partial_{xy} & -\frac{\lambda}{\rho} \partial_{xz} & \frac{1}{\rho} \partial_x & 0 \\ -\frac{\lambda}{\rho} \partial_{xy} & Q_{\frac{\mu}{\rho}} - \frac{\lambda}{\rho} \partial_{yy} & -\frac{\lambda}{\rho} \partial_{yz} & \frac{1}{\rho} \partial_y & 0 \\ -\frac{\lambda}{\rho} \partial_{xz} & -\frac{\lambda}{\rho} \partial_{yz} & Q_{\frac{\mu}{\rho}} - \frac{\lambda}{\rho} \partial_{zz} & \frac{1}{\rho} \partial_z & 0 \\ \rho c^2 \partial_x & \rho c^2 \partial_y & \rho c^2 \partial_z & Q & (1-\gamma)\kappa\Delta \\ \frac{c^2}{\gamma} \partial_x & \frac{c^2}{\gamma} \partial_y & \frac{c^2}{\gamma} \partial_z & 0 & Q_{\frac{\lambda}{\rho}} \end{bmatrix}, \quad (12)$$

$$\det \mathbf{L} = Q_{\frac{\mu}{\rho}}^2 \left[\frac{\kappa c^2}{\gamma \rho} \Delta^2 + Q(-c^2 \Delta + \frac{\kappa(\lambda+\mu)}{\rho^2} \Delta^2) - Q^2 \frac{\kappa+\lambda+\mu}{\rho} \Delta + Q^3 \right], \quad (13)$$

where, nondimensionalizing by density and sound speed and applying Stokes hypothesis for the bulk viscosity term, the coefficients become $\mu/\rho = M_\infty/(\rho \text{Re})$, $\kappa = M_\infty \gamma/(\text{Re Pr})$, and $\hat{\lambda} = \lambda + \mu = \mu/3$. M_∞ is the free stream Mach number, and Re and Pr are Reynolds and Prandtl numbers respectively. Instead of devising a suitable relaxation method for the complicated scalar factor in the brackets of (13), one can opt to a distributed relaxation partially decoupling the linear system associated with operator \mathbf{L} (12). In particular, the distribution matrix

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & -\frac{1}{\rho} \partial_x & 0 \\ 0 & 1 & 0 & -\frac{1}{\rho} \partial_y & 0 \\ 0 & 0 & 1 & -\frac{1}{\rho} \partial_z & 0 \\ \hat{\lambda} \partial_x & \hat{\lambda} \partial_y & \hat{\lambda} \partial_z & Q_{\frac{\lambda+\mu}{\rho}} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (14)$$

results in

$$\mathbf{L}\mathbf{M} = \begin{bmatrix} Q_{\frac{\mu}{\rho}} & 0 & 0 & 0 & 0 \\ 0 & Q_{\frac{\mu}{\rho}} & 0 & 0 & 0 \\ 0 & 0 & Q_{\frac{\mu}{\rho}} & 0 & 0 \\ \mathcal{P} \partial_x & \mathcal{P} \partial_y & \mathcal{P} \partial_z & Q Q_{\frac{\lambda+\mu}{\rho}} c^2 \Delta & (1-\gamma)\kappa\Delta \\ \frac{c^2}{\gamma} \partial_x & \frac{c^2}{\gamma} \partial_y & \frac{c^2}{\gamma} \partial_z & -\frac{c^2}{\gamma \rho} \Delta & Q_{\frac{\lambda}{\rho}} \end{bmatrix}, \quad (15)$$

where $\mathcal{P} \equiv \rho c^2 + \hat{\lambda} Q$. The last two equations remain coupled, requiring a block 2-by-2 matrix solution. This distributed relaxation scheme is still much less expensive than direct relaxation of matrix \mathbf{L} requiring solution for a block 5-by-5 matrix.

Relaxation of Scalar Factors

Efficiency of the distributed relaxation schemes outlined in the previous section is determined by the efficiency of the relaxation (solution) schemes for scalar factors appearing at the main diagonal of the matrices $\mathbf{L}\mathbf{M}$.

For uniformly elliptic operators such as Laplacian, diffusion-dominated convection-diffusion operator, and subsonic full-potential operator many efficient relaxation techniques are available (see textbooks^{3,6,8,9}). For such operators, an important relaxation requirement is efficient reduction of high-frequency errors. All the smooth components are well

approximated on coarse grids built by standard (full) coarsening; therefore, the coarse-grid correction is efficient in reduction of smooth errors.

For nonelliptic and weakly elliptic factors, e.g., convection, convection-dominated convection-diffusion, transonic and supersonic full-potential operators, (smooth) characteristic components cannot be approximated with standard multigrid methods.^{11-13,16,30}

Several approaches aimed at curing the characteristic-component problem have been studied in the literature. These approaches fall into two categories: (1) development of a suitable relaxation scheme (single-grid method) to eliminate not only high-frequency error components but the characteristic error components as well; (2) devising an adjusted coarse-grid operator to approximate well the fine-grid characteristic error components.

Single-Grid Methods

Downstream marching

For hyperbolic problems, the simplest first-category method is downstream marching. If the corresponding discretization is a stable upwind discretization and the characteristic field does not recirculate, then downstream marching is a very efficient solver that yields an accurate solution to a *nonlinear* hyperbolic equation in just a few sweeps (a single downstream sweep provides the exact solution to a linearized problem). The downstream marching technique was successfully applied in solving many CFD problems associated with non-recirculating flows (see, e.g.,^{12,22,25,27,28}). However, if a discretization operator is not fully upwind (e.g., is only upwind biased), straightforward downstream marching is unstable. For the schemes that cannot be directly marched, there are two possible alternatives (also of marching type): defect-correction and predictor-corrector methods.

Defect Correction

Let us consider a defect correction method for a discretized hyperbolic equation

$$L^h u_{i_1, i_2} = f_{i_1, i_2}, \quad (16)$$

with specified inflow boundary conditions u_{0, i_2} .

Let \tilde{u}_{i_1, i_2} be the current solution approximation. Then the improved approximation \bar{u}_{i_1, i_2} is calculated by defect-correction scheme in the following two steps:

1. The correction v_{i_1, i_2} is calculated by solving operator L_d^h with a right-hand side represented by the residual of (16) computed for the current approximation \tilde{u}_{i_1, i_2} . The inflow boundary conditions for v are initialized with the zero values.

$$L_d^h v_{i_1, i_2} = f_{i_1, i_2} - L^h \tilde{u}_{i_1, i_2}. \quad (17)$$

2. The current approximation is corrected as

$$\bar{u}_{i_1, i_2} = \tilde{u}_{i_1, i_2} + v_{i_1, i_2}. \quad (18)$$

The operator L_d^h is called the *driver* operator. It is chosen to be easily solvable and usually less accurate than the target operator L^h ; the latter can be very general. If the iteration converges, steps (17) and (18) can be repeated until the desired accuracy is reached. Usually the efficiency of defect-correction methods is quite satisfactory,^{27,28,31-33} even though in principle the convergence rate of a defect-correction method for nonelliptic operator is normally mesh-size dependent,³⁴⁻³⁶ as explained below.

In several papers (e.g.,^{32,37}), authors studying the defect-correction method for nonelliptic problems observed a slow convergence or even a divergence in some common error norms for the initial iterations and good asymptotic convergence rates afterward. This behavior is different from that observed in solving elliptic problems by the defect-correction method, where the asymptotic convergence rate is the slowest one. This nonelliptic feature is explained by some properties associated with the cross-characteristic interaction (e.g., dissipation and/or dispersion) in the operators involved in the defect-correction iterations. Specifically, this cross-characteristic interaction defines the *penetration distance* (also termed "survival distance"¹²) of a characteristic component. The penetration distance is the distance from the inflow boundary within which the discrete solution of the homogeneous problem reasonably approximates the continuous one (i.e., the discretization error is substantially smaller than the solution).

The penetration distance of a characteristic component is roughly proportional to $\omega^{-1}(\omega h)^{-\frac{p}{q}}$, where q is the highest order of differentiation in the hyperbolic operator under considerations, p is the discrete-operator approximation order, ω is the cross-characteristic frequency of the characteristic component, and h is the mesh size. The ratio of penetration distances of the operators L^h and L_d^h is an important factor for determining the number of defect-correction sweeps required to reduce the algebraic error to the discretization-error level or to reach the asymptotic convergence regime.

When the operators L^h and L_d^h have the same approximation order ($p = r$), efficiency of the defect-correction method is optimal and mesh-size h -independent. If however the operators L^h and L_d^h have different approximation orders (p and r , respectively, $p > r$), then efficiency of the defect-correction method is h -dependent; i.e., the maximal number of sweeps which might be required to reduce the algebraic error to the discretization-error level (or to reach the asymptotic convergence rates) is larger on fine grids than on coarse grids. This is because one has to iterate L_d^h as many times as needed to attain accuracy up to the L^h penetration distance. The worst (largest) ratio of penetration distances is obtained for characteristic components for which the penetration distance in the

target p -order accurate discretization approaches the depth R of the computational domain in the characteristic direction. It follows that the required number of iterations is proportional to $\left(\frac{R}{h}\right)^{\frac{p-r}{p+q}}$.

Predictor-Corrector

One potentially efficient but yet unexploited method to overcome grid-dependent convergence experienced in defect-correction iterations is the predictor-corrector technique. A detailed look into the defect-correction iteration reveals that the computational work distribution is unbalanced: (1) Driver operator iterations at locations beyond the penetration distance do not improve the solution approximation. (2) In successive iterations, the solution approximation near the inflow boundary becomes much more accurate than in the interior; the computational efforts spent in this regions could be more profitably invested at the accuracy frontier.

The predictor-corrector method has been extensively used for ordinary and time-dependent differential equations;^{38,39} however, applications for steady-state nonelliptic problems have been very limited. In predictor-corrector schemes, the final update of the solution at a particular point is computed from the local solution of the target operator. The solution values at downstream points included in the target-operator stencil are predicted from the solution of the driver (predictor) operator. In order to define a family of predictor-corrector schemes, one can divide the computational domain into several time-like layers; the first layer contains all the grid points adjacent to the inflow boundary. Each next layer is composed of the grid points that contribute to the stencils of target operators defined at the points of the previous layer and do not belong to any of the previous layers.

Now, a family of predictor-corrector schemes for solving the correction equation

$$L^h v_{i_1, i_2} = R_{i_1, i_2}^h \equiv f_{i_1, i_2} - L^h \tilde{u}_{i_1, i_2}, \quad (19)$$

where L^h is the target operator, \tilde{u}_{i_1, i_2} is the current solution approximation with residual R_{i_1, i_2}^h , and v_{i_1, i_2} is the desired correction function, can be defined as

PC_0 : The solution of (19) is approximated by solving

$$L_d^h v_{i_1, i_2} = R_{i_1, i_2}^h. \quad (20)$$

This scheme is identical with the defect-correction scheme.

PC_k : Recursive definition of the derived predictor-corrector schemes (recursion with respect to k) can be done as follows: Assume the $(j-1)$ -th layer have already been finally updated in the current sweep. Then, to calculate new values at the next j -th layer one has to perform the following three steps:

1. To predict values at the j -th layer with PC_{k-1} scheme.
2. To predict values at the $(j+1)$ -th layer with PC_{k-1} scheme.
3. To update values at the j -th layer by directly relaxing (19).

The schemes for $k = 0, 1, 2$ have been tested for a linearized supersonic full-potential operator.¹⁷

Multigrid methods

Recirculation

Downstream marching methods are not viable for problems with closed characteristics. Alternative discretization and solution techniques should be considered. The discretization issue becomes especially important for flows with streamlines that do not start and end at boundaries, but constitute closed curves. In such cases, even a very small viscosity plays an important role in determining the flow throughout the domain. The solution in the limit of vanishing viscosity depends very strongly on how the viscosity coefficients tend to zero. The propagation of information from the boundary into the domain is governed by the viscous terms no matter how small they may be. It has been shown⁴⁰ for both the scalar convection-diffusion problem and the incompressible Navier-Stokes equations that varying cross-stream numerical viscosity (caused usually by varying angles between the stream and the grid lines; e.g., in standard upwind and upwind biased schemes) may prevent convergence to a physically realizable solution. In the most general case, it can be shown that even isotropic viscosity is not sufficient for convergence to a physical solution, and one must actually specify a *uniform* viscosity. However, for the homogeneous problems there are several indications^{40,41} (though no proof) that isotropy suffices.

To obtain a discretization scheme that exhibits the appropriate physical-like behavior for vanishing viscosity, one must either add sufficient explicit isotropic viscosity that will dominate the anisotropic numerical viscosity of the convection operator, or else derive a discrete convection operator with numerical viscosity satisfying the condition of isotropy. An upwind isotropic-viscosity discretization has been derived.⁴¹

One general approach to the algebraic solution of nonelliptic equations with closed characteristics is to apply a multigrid method with an overweighted upwind-biased residual restriction. Efficient multigrid solvers for recirculating convection equation have already been demonstrated.^{41,42} This approach is well combined with the distributed relaxation method for the RANS equations, because within a distributed relaxation sweep a multigrid solver with optimal overweighting can be applied to a separate scalar nonelliptic equation with closed characteristics.

Another solution approach is to apply some general techniques to approximate indirectly smooth characteristic components. Among helpful techniques are recombination of iterants, cycles with high indexes, and implicit alternative-direction relaxation. Recombination of iterants (solution approximations on different stages of a multigrid algorithm) at each grid level eliminates several (number of iterants minus one) error components, those, more specifically, that are most prominent in the residual function. Making increasingly many coarse-grid iterations per each fine-grid iteration, cycles with high indexes solve the characteristic-component problem on coarser grids. Implicit alternative-direction relaxation simulates downstream marching in the regions with open characteristics and efficiently transfers information in the regions with characteristics closely aligned with the grid. Theoretically, each of these methods cannot completely resolve the problem of poor coarse-grid correction to the fine-grid smooth characteristic error components. The problem already manifests itself in two-level algorithms with any type of local relaxation. On fine grids the number of problematic error components may increase, and many cycles may be needed to collect the necessary number of the fine-grid iterants to exclude all the troubling error components. However, it has been shown experimentally⁴³ that a combination of implicit alternative-direction defect-correction type relaxation, recombination of iterants on all the levels, and W-cycles can result in a relatively efficient multigrid solver for recirculating flow problems on practical grids.

Full-Potential Operator

The full-potential operator is a variable type operator, and its solution requires different procedures in subsonic, transonic, and supersonic regions. In deep subsonic regions, the full-potential operator is uniformly elliptic and therefore standard multigrid methods yield optimal efficiency. When the Mach number approaches unity, the operator becomes increasingly anisotropic and, because smooth characteristic error components cannot be approximated adequately on coarse grids, classical multigrid methods severely degrade. In the deep supersonic regions, the full-potential operator is uniformly hyperbolic with the stream direction serving as the time-like direction. In this region, an efficient solver can be obtained with a downstream marching method. However, downstream marching becomes problematic for the Mach number dropping towards unity, because the Courant number associated with this method becomes large. Thus, a special procedure is required to provide an efficient solution for transonic regions. A possible local procedure^{13,15,17,30} is based on piecewise semicoarsening and some rules for adding dissipation at the coarse-grid levels.

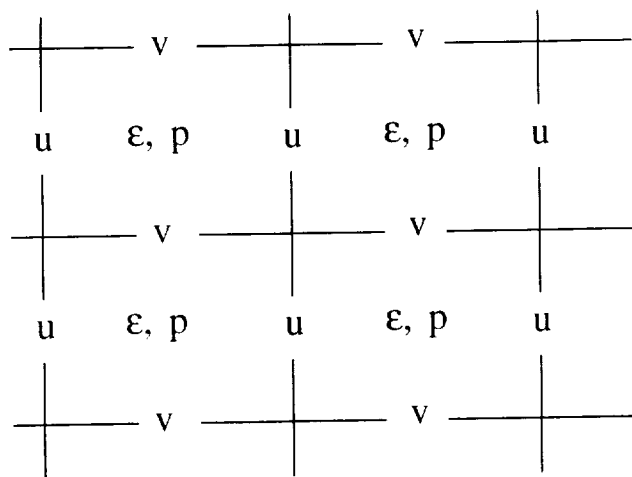


Fig. 1 Staggered arrangement of primitive variables for Navier-Stokes discretization.

A similar technique can be used to construct an efficient marching-free multigrid solver for convection-dominated equations. This method¹⁶ employs a colored relaxation scheme and is very attractive for massive parallel computing. A highly parallel multigrid solver for the supersonic full-potential operator may be obtained by methods similar to the wave/ray multigrid.¹⁴

Discrete Equations

Traditional Factorizable Schemes

As mentioned in the introduction, factorizability of a target discrete scheme significantly simplifies the distributed relaxation design. The main obstacle in this case to efficient solution is that the discretizations obtained for the scalar factors in the discrete determinant are not always convenient. The search for new factorizable discretization schemes (see Summary of Recent Progress below) is chiefly motivated by the need to derive discrete schemes with the resulting discretizations of scalar factors satisfying some desired properties (e.g., correct alignment with the physical anisotropies, compactness, availability of an efficient relaxation scheme, etc.).

Staggered-Grid Discretization for Navier-Stokes Equations.

The staggered-grid discretization dating back to the mid 60's⁴⁴⁻⁴⁶ is one of the first factorizable discretizations for incompressible flow equations. Compressible flow discretizations with a staggered arrangement of variables have also been studied.^{3, 27, 47} A usual placement of primitive variables in two dimensions is depicted in Figure 1. With this staggering, (a) the off-diagonal first derivatives in (3), (8), and (12) can be approximated as short central differences; (b) the second derivatives in (12) can be compositions of corresponding central first derivatives; (c) the convection-diffusion operators, Q_ν , can be approximated by any proper discretizations Q_ν^h . For discrete factorizability,

it is important to have the same discretization for each of the Q_ν -operators in the momentum equations; the convection-diffusion operators in other equations can be different. The convection terms in the momentum and energy equations are usually upwind or upwind-biased; for simplicity, below we assume that all these terms are the same. With such differencing, the discrete schemes mimic the factorizability property of the differential equations, and the discrete system determinants can be factored as $\det \mathbf{L}^h = (Q_\nu^h)^2 \Delta^h$ (incompressible Navier-Stokes) or $\det \mathbf{L}^h = (Q^h)^3 (Q^h \bar{Q}^h - c^2 \Delta^h)$ (compressible Euler), where Δ^h in three dimensions is the seven-point h -Laplacian, Q^h is an upwind or upwind biased discretization of the convection operators in the momentum and energy equations, \bar{Q}^h is a convection-operator discretization for the pressure term in the fourth equation of (8), hence $Q^h \bar{Q}^h - c^2 \Delta^h$ is a discrete approximation to the full-potential operator. The discrete determinant computed for the compressible Navier-Stokes equations is similar to the differential determinant (13).

The discrete distribution matrices follow directly from the continuous matrices (5), (10), and (14). The short central differences are used for the approximation of all the off-diagonal first derivatives; the convection parts in the Q -operators are the same as those in the momentum equations. The resulting products $\mathbf{L}^h \mathbf{M}^h$ are similar to those for the continuous problems with the main diagonals composed of the factors of the discrete determinants.

Distributed-relaxation solvers have been successfully applied to the staggered-grid discretization schemes for subsonic compressible²⁷ and incompressible^{12, 28} flow problems.

In computing the Euler system of equations, the main disadvantages of the staggered-grid scheme relate to the discrete stencil of the full-potential operator. For subsonic flow problems, the downwind differencing applied for the \bar{Q}^h term results in a full-potential-operator stencil that is somewhat wide (because of the $Q^h \bar{Q}^h$ term) and poorly aligns with the physical (cross-stream) anisotropies in approaching the transonic regime. For supersonic flow, where the problem is purely hyperbolic, the stencil is not fully upwind (even if the \bar{Q}^h term is upwind differenced) implying more involved marching schemes.

Recently, a new approach to building discretization schemes that allows *any* desired differencing for the full-potential factor of the system determinant without compromising the scheme factorizability has been discovered. This approach is discussed subsequently in application to central collocated-grid discretizations (see also¹⁸), but it applies to staggered grids as well.

Collocated-Grid Discretizations for the Euler Equations.

Another example of a factorizable scheme is a collocated-grid scheme with the second-order central differencing for the off-diagonal first derivatives in (3), (8), and (12). The convection operators in the momentum and energy equations are again upwind or upwind biased; the differencing of the convection term applied to the pressure may alternate from downwind (downwind-biased) in subsonic mode to upwind (upwind-biased) in supersonic mode.

A typical difficulty associated with this type of schemes is a poor measure of h -ellipticity in the discrete approximation for the full-potential factor of the system determinant. To be more specific, let us define the collocated-grid discretization \mathbf{L}^h of the matrix operator (8) as

$$\mathbf{L}^h = \begin{bmatrix} Q^h & 0 & 0 & \frac{1}{\rho} \partial_x^{2h} & 0 \\ 0 & Q^h & 0 & \frac{1}{\rho} \partial_y^{2h} & 0 \\ 0 & 0 & Q^h & \frac{1}{\rho} \partial_z^{2h} & 0 \\ \rho c^2 \partial_x^{2h} & \rho c^2 \partial_y^{2h} & \rho c^2 \partial_z^{2h} & \bar{Q}^h & 0 \\ \frac{c^2}{\gamma} \partial_x^{2h} & \frac{c^2}{\gamma} \partial_y^{2h} & \frac{c^2}{\gamma} \partial_z^{2h} & 0 & Q^h \end{bmatrix}, \quad (21)$$

where the discrete derivatives, $\partial_x^{2h}, \partial_y^{2h}, \partial_z^{2h}$, in all off-diagonal terms are the wide (with mesh spacing $2h$) second-order-accurate central-differencing approximations. All the diagonal terms, Q^h , except \bar{Q}^h in the fourth equation, are discretized with the same second-order-accurate upwind (or upwind-biased) discretization scheme. In the subsonic regime ($|\mathbf{u}|^2 = \bar{u}^2 + \bar{v}^2 + \bar{w}^2 < c^2$), the \bar{Q}^h -term is discretized with a second-order-accurate downwind (or downwind-biased) discretization. The determinant of the matrix operator \mathbf{L}^h is given by

$$(Q^h)^3 [Q^h \bar{Q}^h - c^2 \Delta^{2h}], \quad (22)$$

where Δ^{2h} is a wide discretization of the Laplace operator. The full-potential-operator approximation appearing in the brackets has two major drawbacks: (1) For slow velocities ($|\mathbf{u}| \ll c$), the discrete operator is dominated by the non- h -elliptic wide Laplacian, and efficiency of *any* local relaxation severely degrades. (2) For near-sonic regimes (the Mach number $M = |\mathbf{u}|/c \approx 1$), the discrete operator stencil does not reflect the physical anisotropies of the differential full-potential operator; the discrete operator exhibits a very strong coupling in the streamwise direction, while the differential operator has strong coupling only the cross-stream directions.

Several approaches to cure the lack of h -ellipticity (mainly in applications to incompressible-flow equations) have been proposed in the literature (e.g.,^{49,50}). Some of the approaches are associated with introduction of additional terms increasing the measure of

h -ellipticity in the system of equations, others propose averaging (filtering) spurious oscillations. The problem of wrong anisotropies in the full-potential-operator has not been sufficiently investigated. In two dimensions, it is possible to construct a discretization that satisfies the following properties: (1) At low Mach numbers, the discretization is dominated by the standard (with mesh spacing h) h -elliptic Laplacian. (2) For the transonic Mach numbers, the discretization tends to the optimal discretization^{13,17,30} for the sonic-flow full-potential operator. (3) For supersonic Mach numbers, the discretization becomes upwind (upwind-biased) and can be solved by marching. The problem of constructing a good high-order discretization for the transonic full-potential operator in three dimensions is still open.

Non-Factorizable schemes

The majority of discrete schemes in current use, especially for compressible flow but also more recently for incompressible flow, are based upon a flux-splitting approach. The basis of this approach is the solution of the Riemann problem (i.e., the time evolution of two regions of flow initially separated by a diaphragm) applied on a dimension by dimension basis. This methodology has enabled the robust treatment of flows with strong shocks and complex geometries. However, the derived schemes are not discretely factorizable, except in one dimension.

These discrete equations have always been solved using collective relaxation (or pseudo-time-stepping) in multidimensional multigrid algorithms. A better efficiency should be realizable with a relaxation scheme that efficiently reduces both the high-frequency and characteristic error components. Such a scheme should combine two different relaxation methods: (1) A local relaxation scheme treating directly the conservation equations and reducing the high-frequency error components. (2) For reduction of characteristic error components, a defect-correction (or predictor-corrector) method with a factorizable driver (predictor). This approach has not been tried as yet.

Boundary Conditions and Discontinuities

Boundaries and discontinuities introduce some additional complexity in distributed relaxation. The determinant of $\mathbf{L} \mathbf{M}$ is usually higher order than the determinant of \mathbf{L} . Thus, as a set of new variables, $\delta \mathbf{w}$ would generally need additional boundary conditions. In relaxation, because the ghost variables can be added in the external part of the domain, it is usually possible to determine suitable boundary conditions for $\delta \mathbf{w}$ that satisfy the original boundary conditions for the primitive variables. Examples are available²⁸ for incompressible flow with entering and no-slip boundaries. However, to construct such extra boundary

conditions may be difficult and/or time-consuming in general. In addition, enforcing these boundary conditions causes the relaxation equations to be coupled near the boundaries, not decoupled as they are in the interior of the domain.

Thus, near boundaries and discontinuities, the general approach³⁻⁵ is to relax the governing equations directly in terms of primitive variables. Several extra sweeps of robust (but possibly slowly converging) relaxation, such as block-Newton-Kacmarcz relaxation, can be made in these special regions after (and perhaps also before) the distributed relaxation pass to reduce residuals to the average level characterizing the regular flow field. The additional sweeps will not seriously affect the overall complexity because the number of boundary and/or discontinuity points is usually small in comparison with the number of interior points. An example of TME using local relaxation at shocks and boundaries and distributed relaxation over the rest of the domain has been demonstrated.²⁹

Analysis

As mentioned previously, it is important in attaining optimal efficiency to understand all the difficulties that present themselves in application. Analysis methods are quite helpful in this regard, and the main tools are discussed below. In iterative methods solving elliptic problems, the main mechanism of convergence is damping of error components. In solution of hyperbolic scalar equations, there is another very important convergence mechanism: the downstream evolution of the error components. In the presence of this additional mechanism, the accuracy first achieved near the inflow boundary and is then propagated into the interior of the domain.

The recognition of this additional convergence mechanism urges modifications in the standard analysis developed for elliptic problems. Basically, one can distinguish four types of analysis applied to nonelliptic problems: (1) standard linear-algebraic matrix analysis, (2) modified zero-mode-exclusion full-space Fourier mode analysis, (3) half-space analysis of the first differential approximations (FDA),^{11,51,52} and (4) discrete half-space analysis. Briefly, the first differential approximation (also called modified equation) to a difference operator on a grid with mesh size h is the Taylor expansion of this difference operator in terms of h truncated to the first terms including the least nonzero power of h . The quality of an analysis applied to nonelliptic problems is determined by how well the analysis handles the characteristic components.

Matrix analysis

The most general and precise analysis methods are the linear-algebra matrix analysis methods applied to the corresponding linearized problem. This analysis considers the difference operators without assump-

tions about the solution and boundary conditions. It can be applied to variable-coefficient problems as well. This analysis was found very useful for analyzing one-dimensional problems. However, the enormous computational complexity of this analysis makes it not viable for multidimensional problems. Although, the analysis complexity can be reduced considerably by assuming Fourier modes in two of the three spatial directions.

Modified full-space Fourier mode analysis

The modified full-space Fourier mode analysis is a modification of the standard full-space Fourier mode analysis excluding from the consideration all the zero modes (the modes with vanishing symbols).⁹ It is the simplest and most popular type of analysis (e.g., see applications in^{32,33}). This analysis estimates only the amplification (damping) factor. Its inherent disadvantage is the inability to take the influence of the inflow boundary into account. This explains its failure in describing the downstream error evolution. However, the modified full-space analysis can also be useful for analyzing the effect of forcing terms.

FDA half-space analysis

The FDA half-space analysis is a relatively simple and efficient tool for analyzing the effect of the inflow boundary. Examples of applications of this analysis are available.^{11,12,51,52} The first differential approximations are considered on a half-space including cut by an inflow boundary. The boundary conditions are represented by one Fourier mode at a time. The FDA analysis provides a good *qualitative* description of the downstream error evolution. This analysis focuses on characteristic components and, therefore, considers homogeneous problems. Note that a combination of the FDA analysis with the modified full-space analysis can provide a good insight for nonhomogeneous problems as well. The disadvantages of this analysis are the inability to provide quantitative estimates, to analyze the effect of different boundary condition discretizations, and to address the asymptotic convergence rate.

Discrete half-space analysis

The discrete half-space analysis^{30,35,36} considers the discretizations in their exact form rather than their differential approximation, while the boundary data are represented by a Fourier component. This analysis translates the original multidimensional problem into a one-dimensional discrete problem, where the frequencies of the boundary Fourier components are considered as parameters. To regularize the half-space problem, the solution is not allowed to grow faster than a polynomial function. This tool is very accurate; it can be used to explain in detail many phenomena observed in solving nonelliptic equations and provides a close prediction of the actual solution behavior.

The one-dimensional solution obtained in the dis-

crete half-space analysis has two different representation forms: (1) away from the boundary, the solution is defined as a linear combination of a finite number of analytical components; this region is called the *analytical representation region*; (2) in the region adjacent to the inflow boundary, the solution is defined pointwise; this zone is referred to as the *pointwise representation region*. With each further iteration described by the analysis, the pointwise representation region penetrates by a finite number of mesh sizes into the interior. By using these representations, the computational complexity of the analysis becomes much less than that associated with the one-dimensional matrix analysis. In the asymptotic regime, when the pointwise representation zone covers all the domain, this analysis becomes a discrete one-dimensional matrix analysis of the multidimensional problem.

The discrete half-space analysis provides a *quantitative* description of the approximate solution; it predicts the convergence rate for each iteration and the asymptotic convergence rate. It can be easily adjusted to analyze the global effect of *any* local discretization of the inflow boundary conditions. This adjustment can be done just by widening the initial pointwise representation region at the inflow boundary. If necessary, the analysis can take into account the influence of the discretized *outflow* boundary conditions as well. Generally, this discrete half-space analysis treats completely both mechanisms of convergence, damping and downstream evolution of errors, associated with nonelliptic problem solvers.

Summary of Recent Progress

The first TME solver applying the distributed relaxation approach for solution of a free-stream incompressible Navier-Stokes equations has been demonstrated long ago.¹² Recently, TME has been achieved for high-Reynolds-number incompressible wake flow and the boundary layer flow associated with a finite flat plate.²⁸ An initial extension of this work to compressible (subsonic) viscous flow has also been completed.²⁷ In all these calculations, a staggered arrangement of variables on Cartesian grids has been used. With distributed relaxation, the system of equations has been decomposed (i.e., factored) everywhere, except near boundaries where the equations remained coupled. Two-dimensional FMG solvers with just one multigrid cycle per grid level and a total computational work equivalent to about 10 target-grid residual evaluations converged the drag to the discretization accuracy.

Recently, a new multidimensional factorizable scheme for the Euler equations has been developed⁵³ for Cartesian coordinates and extended through generalized coordinates to external lifting flows around airfoils with both subcritical and supercritical freestream Mach numbers.^{54,55} This scheme is the first flux-

difference-splitting discretization factorizable in multiple dimensions. The starting point for the scheme is the first-order discretization of the flux-difference splitting scheme of Roe. Correction terms are added in the form of mixed derivatives to make the scheme both second-order accurate and discretely factorizable. The resulting scheme is second-order accurate and compact in comparison to other scheme. Discrete factorizability is achieved by using some non-standard wide approximations for spatial derivatives to ensure that the identities

$$\begin{aligned}\partial_{xx}\partial_{yy} &= \partial_{xy}\partial_{xy}, \\ \partial_{xx}\partial_y &= \partial_{xy}\partial_x, \\ \partial_{yy}\partial_x &= \partial_{xy}\partial_y\end{aligned}$$

are satisfied on the discrete level. The determinant of the resulting scheme is composed of an upwind differenced convection factor and an h -elliptic approximation for the full-potential factor. The distributed relaxation is possible by using a left and right distribution matrix, although this has not been applied as yet.

In numerical tests performed for this scheme, the multigrid solver employed alternating-direction collective Gauss-Seidel relaxation. The alternating-direction relaxation is necessary since the full-potential factor is not separately treated. Computations for subsonic and transonic channel flows with essentially grid-independent convergence rates have been presented.⁵⁴ Grid-independent convergence rates have also been attained for a flow with stagnation points.⁵⁵ The subsonic-flow convergence rates observed in multigrid V-cycles were quite fast (about 0.3 per cycle) and only slightly grid dependent. The rates somewhat deteriorate in transonic/supersonic computations. Further developments of this scheme are presented in two papers at this conference.^{55,56} The scheme applies at low Mach numbers although it has yet to be extended to viscous flows.

Another approach to building factorizable schemes with suitable discretizations for scalar factors has been explored in papers of the second and third authors.^{48,57} The approach is based on a collocated-grid scheme with a mechanism that allows one to improve the h -ellipticity measure by obtaining *any* desired discretizations for the full-potential factor of the system determinant without compromising the discrete factorizability. Also, the distribution matrices follow directly from the discrete forms for \mathbf{M} presented earlier. The same approach can be applied for incompressible-flow problems and to staggered-grid discretizations as well.

The starting point is the discretization (21). The way proposed to improve the discrete full-potential operator is to change the discretization of \bar{Q}^h to $\bar{Q}^h + \mathcal{A}^h$. Then the discrete full-potential operator is changed to

$$Q^h \mathcal{A}^h + Q^h \bar{Q}^h - \epsilon^2 \Delta^{2h}.$$

where $\mathcal{A}^h = (Q^h)^{-1} \mathcal{D}^h$, $\mathcal{D}^h = \mathcal{F}^h - (Q^h \bar{Q}^h - c^2 \Delta^{2h})$, and \mathcal{F}^h is a desired approximation for the full-potential factor. In smooth regions, \mathcal{A}^h is second-order small (proportional to h^2), hence the overall second-order discretization accuracy is not compromised. The operator $(Q^h)^{-1}$ is a nonlocal operator and its introduction can be effected through a new auxiliary variable ψ^h and a new discrete equation $Q^h \psi^h = \mathcal{D}^h p^h$.

Thus, the corrected discrete approximation to (8) is defined as

$$\mathbf{L}^h = \begin{bmatrix} Q^h & 0 & 0 & 0 & \frac{1}{\rho} \partial_x^{2h} & 0 \\ 0 & Q^h & 0 & 0 & \frac{1}{\rho} \partial_y^{2h} & 0 \\ 0 & 0 & Q^h & 0 & \frac{1}{\rho} \partial_z^{2h} & 0 \\ 0 & 0 & 0 & Q^h & -\mathcal{D}^h & 0 \\ \rho c^2 \partial_x^{2h} & \rho c^2 \partial_y^{2h} & \rho c^2 \partial_z^{2h} & 1 & \bar{Q}^h & 0 \\ \frac{c^2}{\gamma} \partial_x^{2h} & \frac{c^2}{\gamma} \partial_y^{2h} & \frac{c^2}{\gamma} \partial_z^{2h} & 0 & 0 & Q^h \end{bmatrix}. \quad (23)$$

The corresponding distribution matrix, \mathbf{M}^h , for distributed relaxation is defined as

$$\mathbf{M}^h = \begin{bmatrix} 1 & 0 & 0 & 0 & -\frac{1}{\rho} \partial_x^{2h} & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{\rho} \partial_y^{2h} & 0 \\ 0 & 0 & 1 & 0 & -\frac{1}{\rho} \partial_z^{2h} & 0 \\ 0 & 0 & 0 & 1 & \mathcal{D}^h & 0 \\ 0 & 0 & 0 & 0 & Q^h & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad (24)$$

so that the resulting matrix $\mathbf{L}^h \mathbf{M}^h$ becomes lower triangular as

$$\mathbf{L}^h \mathbf{M}^h = \begin{bmatrix} Q^h & 0 & 0 & 0 & 0 & 0 \\ 0 & Q^h & 0 & 0 & 0 & 0 \\ 0 & 0 & Q^h & 0 & 0 & 0 \\ 0 & 0 & 0 & Q^h & 0 & 0 \\ \rho c^2 \partial_x^{2h} & \rho c^2 \partial_y^{2h} & \rho c^2 \partial_z^{2h} & 1 & \mathcal{F}^h & 0 \\ \frac{c^2}{\gamma} \partial_x^{2h} & \frac{c^2}{\gamma} \partial_y^{2h} & \frac{c^2}{\gamma} \partial_z^{2h} & 0 & -\frac{c^2}{\gamma \rho} \Delta^{2h} & Q^h \end{bmatrix}. \quad (25)$$

The scheme as defined above is valid for nonconservative flows. A version to be used for distributed relaxation of conservative equations has also been designed.⁵⁷

Concluding Remarks

A general multigrid approach to attain TME solvers for large-scale CFD applications has been outlined. This approach focuses on fast convergence to the solution of the differential equations, not necessarily providing fast asymptotic convergence rates. The considered measure of the method efficiency is convergence to the differential solution, i.e., fast reduction of algebraic errors below the discretization error level on each mesh.

Because the governing equations are a coupled set of nonlinear conservation equations with discontinuities

and singularities, attainment of full efficiency requires that each of three error components be addressed: (1) high-frequency errors (2) uniformly smooth errors and (3) characteristic errors. These errors are reduced through a combination of distributed relaxation and local relaxation at each grid. The relaxations are followed by an FAS correction from a coarser grid where the initial coarse-grid residuals are found from the fine-grid residuals of the conservative equations with (conservative) full-weighting restriction.

The distributed relaxation procedure is designed to reduce errors in the smoothly varying regions of the domain and relies on the factorizability property of the governing differential equations to isolate and treat optimally different factors arising in the determinant of the differential operator. Optimal relaxation of some particular factors may itself involve a separate inner multigrid cycle over a limited subdomain.

Local relaxation is a procedure designed to reduce large residuals of the conservative equations at discontinuities/singularities/boundaries and is applied in these regions as well as in general where the residuals are largest. It employs a robust scheme (e.g., some block relaxation) applied before and after a distributed relaxation sweep.

The particular factors arising in distributed relaxation of the Euler and Navier-Stokes equations have been discussed from the standpoint of the differential and the discrete equations. Methods for relaxing and analyzing these factors within the multigrid context have been presented and evaluated. Recent progress in development textbook-efficient multigrid solvers based on the distributed relaxation approach has been summarized.

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